IPP-QM-11: Contextuality and the BKS theorem

James Read[1]

[1] Faculty of Philosophy, University of Oxford, UK, OX2 6GG

MT25

The course

1. Basic quantum formalism
2. Density operators and entanglement
3. Decoherence
4. The measurement problem
5. Dynamical collapse theories
6. Bohmian mechanics
7. Everettian structure
8. Everettian probability
9. EPR and Bell's theorem
10. The Bell-CHSH inequalities and possible responses
11. Contextuality
12. The PBR theorem
13. Quantum logic
14. QBism
15. Pragmatism and relational quantum mechanics
16. Wavefunction realism

Today

Contextuality

The Bell–Kochen–Specker theorem introduced

The Clifton–Stairs state-dependent proof

Proof of the BKS theorem

The Klyachko–Can–Binicioglu–Shumovsky inequality

Significance of the BKS theorem

Gleason's theorem

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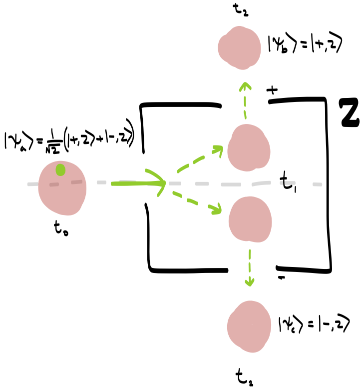
- Different experimental arrangements lead to *different* physics.
- So measurements are not *simply* revealing properties of the system being measured.

Contextuality reminder: Bohmian mechanics



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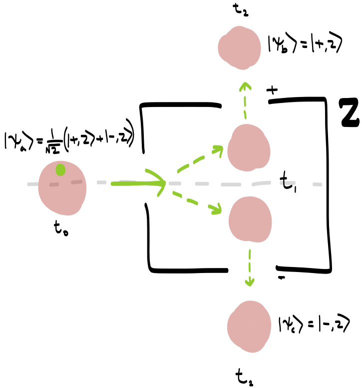

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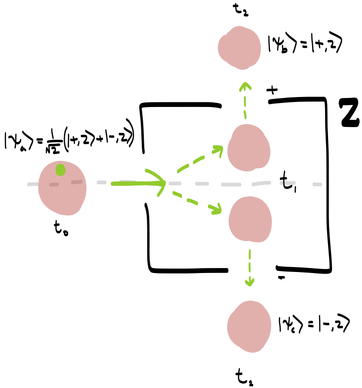

- The Bohm trajectories are deterministic. The outcomes of a measurement are fixed in advance.
- But the outcomes of e.g. Stern–Gerlach experiments also depend upon the context of *how* the measurement is performed.

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- The conjunction of 'outcome determinism' with 'measurement non-contextuality' used to be called 'non-contextuality' or 'non-contextual value definiteness.'
- Now it is sometimes called **traditional non-contextuality** or 'Kochen–Specker non-contextuality'.

Non-contextuality no-go theorems

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- Traditionally non-contextual theories are possible in 2D Hilbert spaces (e.g. the 'Kochen–Specker qbit').
- They are provably not possible in any Hilbert space of larger dimension:
  - The Clifton–Stairs state-dependent proof.
  - The Bell–Kochen–Specker (BKS) state-independent proof.
  - The Klyachko–Can–Binicioglu–Shumovsky (KCBS) inequality.

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- The BKS theorem tells us that any hidden variable theory which assigns values to all properties represented by projectors must be *contextual*.
- Specifically, whether or not a system is found, on measurement, to possess a given property must depend upon which *other* properties are measured simultaneously.
- As Wallace (2007, p. 51) writes, “Contextuality seems well-nigh inconsistent with the idea that systems determinately do or do not possess given properties and that measurements simply determine whether or not they do.”

Today

The Clifton–Stairs state-dependent proof

Before getting to the full BKS contextuality theorem, we'll first look at a simpler, 'state-dependent' proof due to Clifton (1993) and Stairs (1992).

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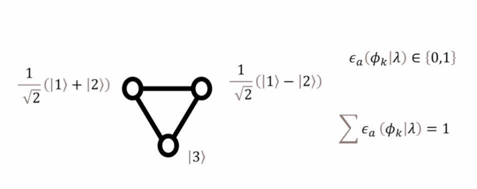

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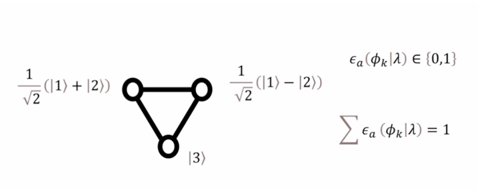

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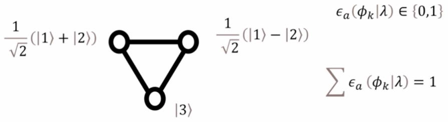

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- We require that the measurement outcomes are 0 or 1, and that they sum to 1.

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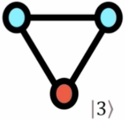
- In the diagrams to follow, red dots indicate where the ontic state lives.

The Clifton–Stairs proof



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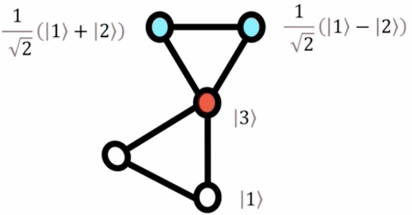

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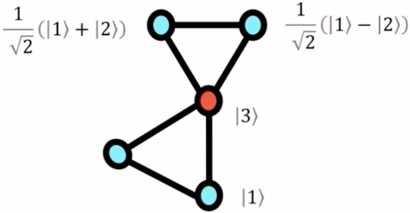
The Clifton–Stairs proof

So, naturally, the other circles are going to have to be blue:



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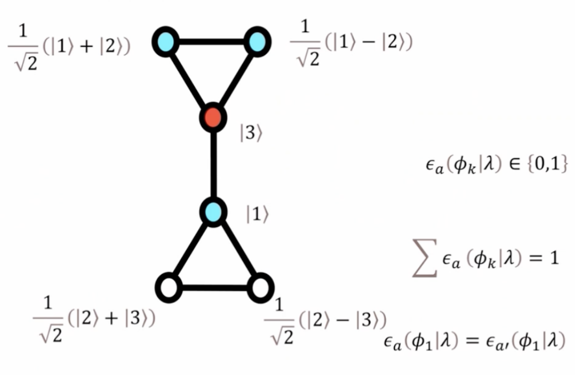
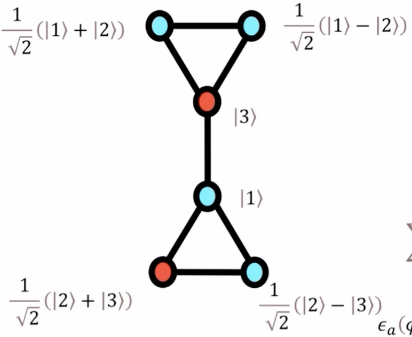
Now consider a scenario like this:



One of the other two circles is going to have to be red...

The Clifton–Stairs proof

E.g.:



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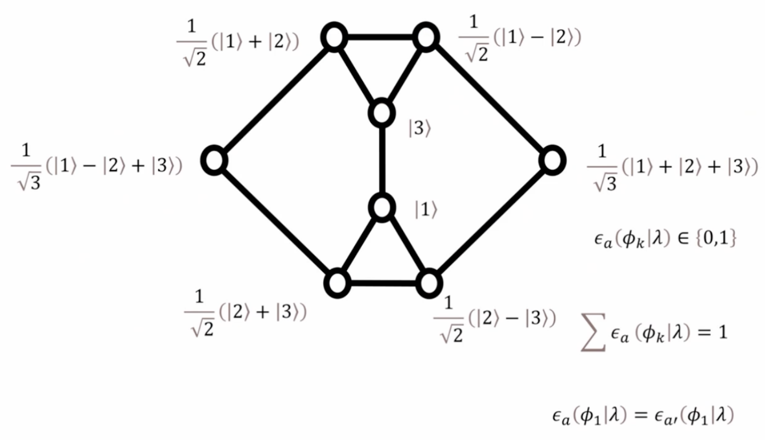
Now consider this scenario:



Notice that the far-left and far-right states are *not orthogonal to each other*.

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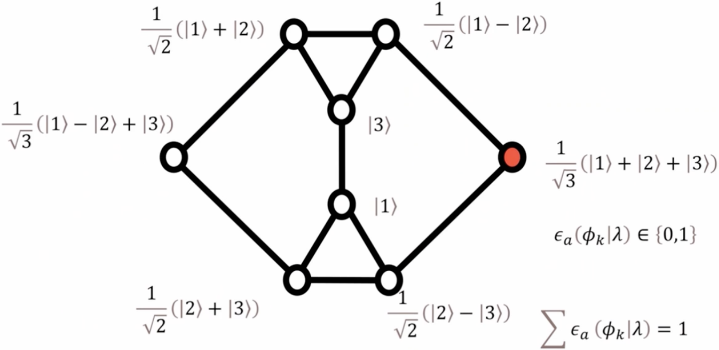

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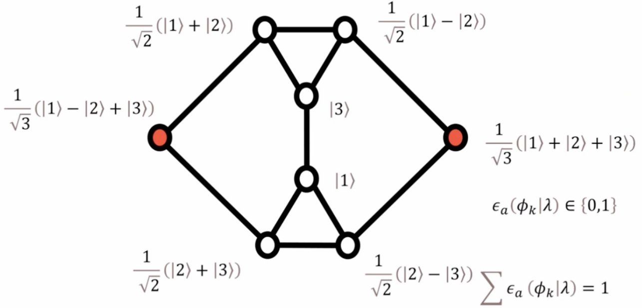
- In that case, we have:



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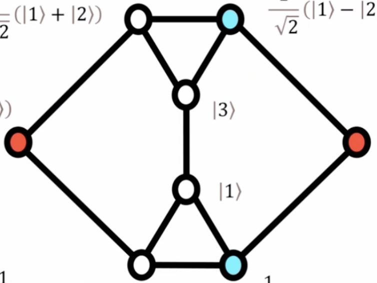

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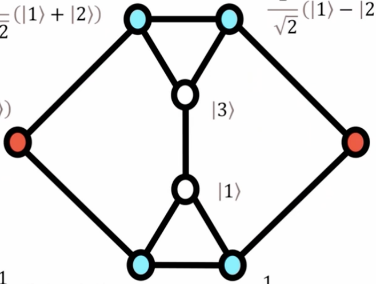

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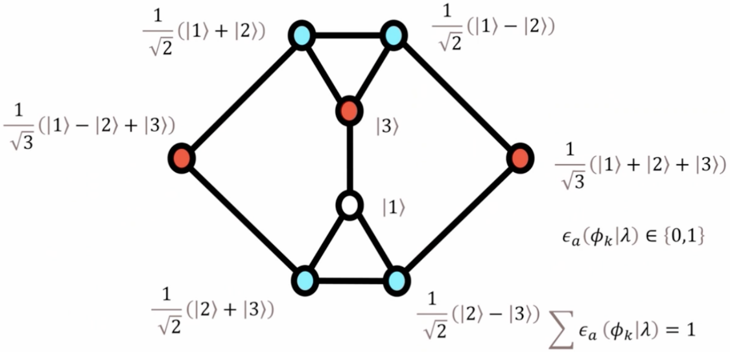

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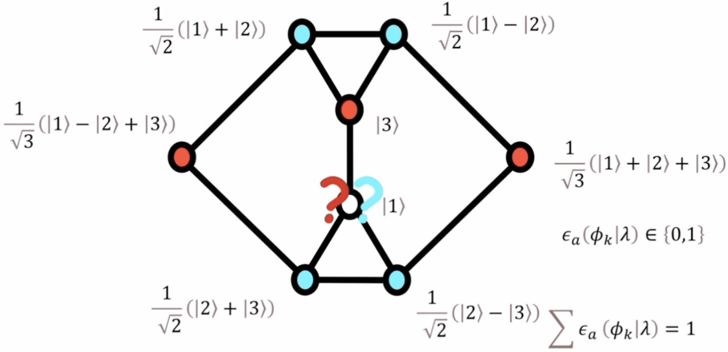

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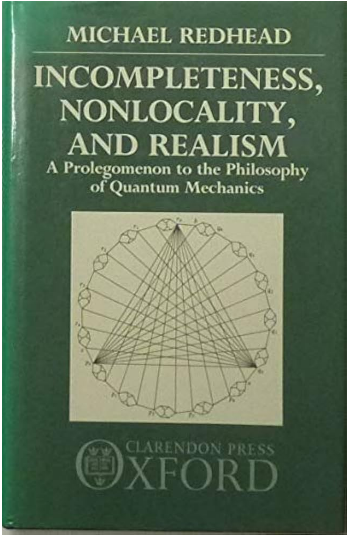

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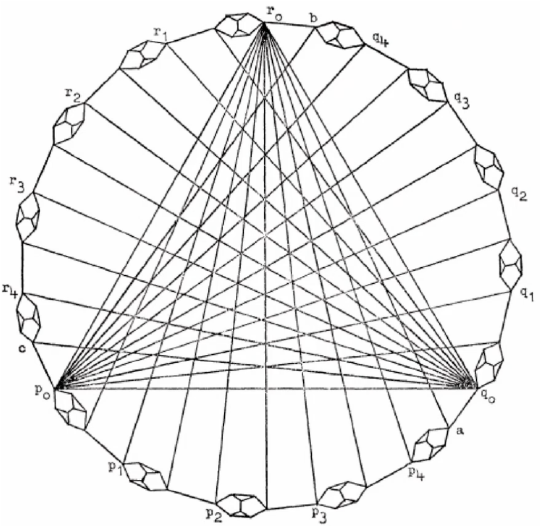

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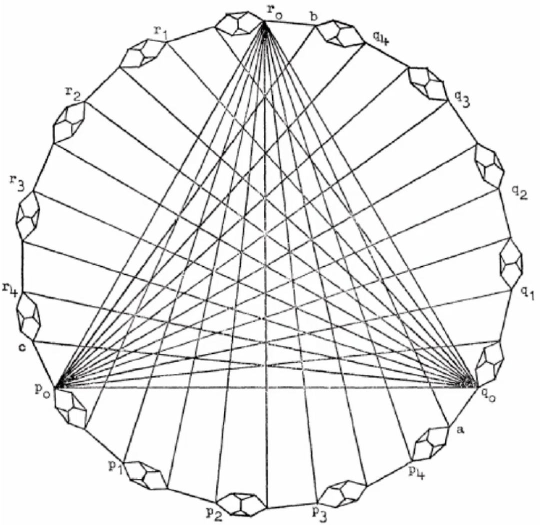

(Note: Stairs was working after Kochen & Specker, and noticed that one could use the mini-diagrams for a state-dependent proof.)

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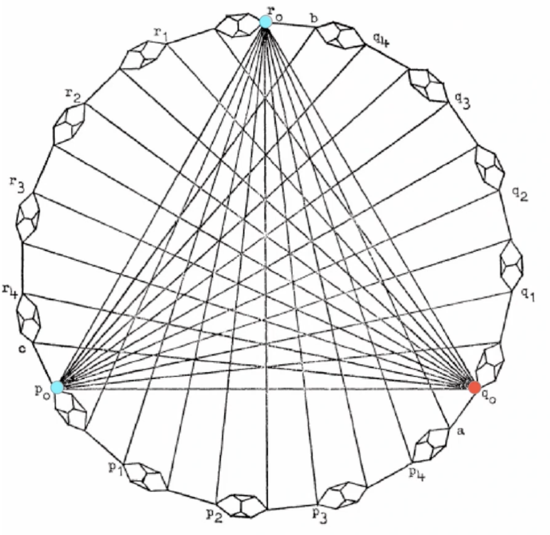

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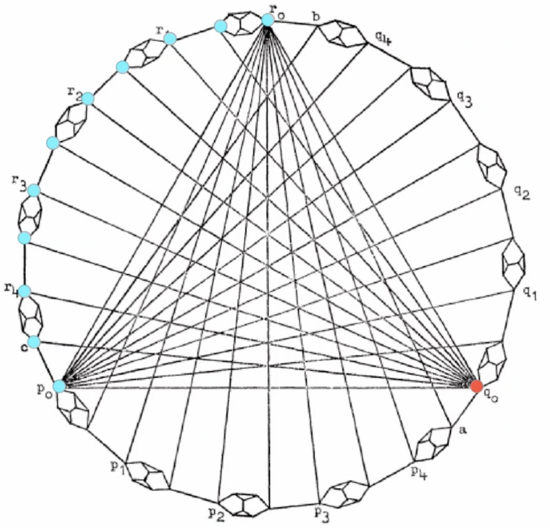

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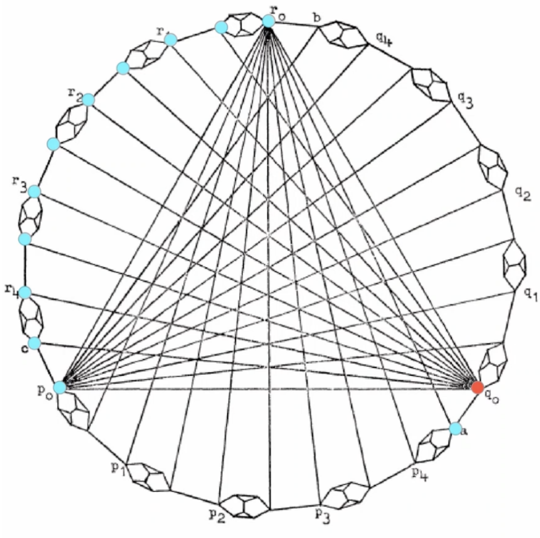

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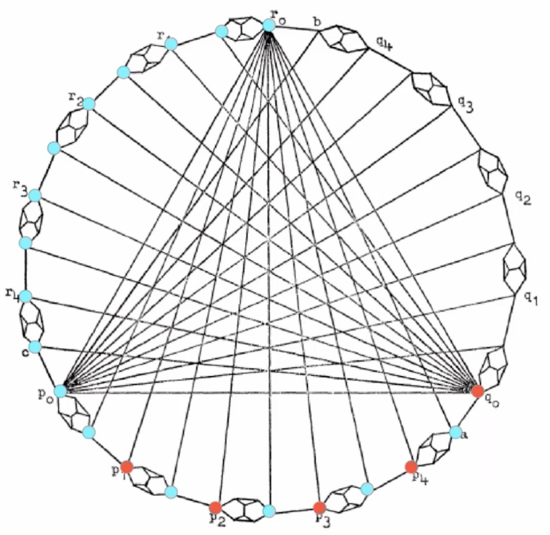

(At least one of the blobs for each of the pairs at the bottom has to be red given that we have a blue at the top.)

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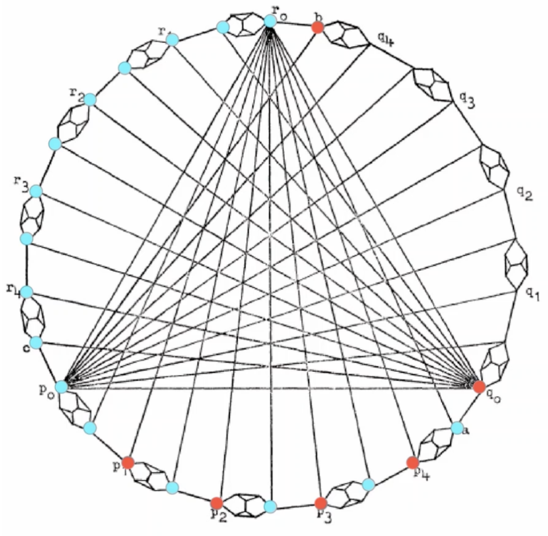

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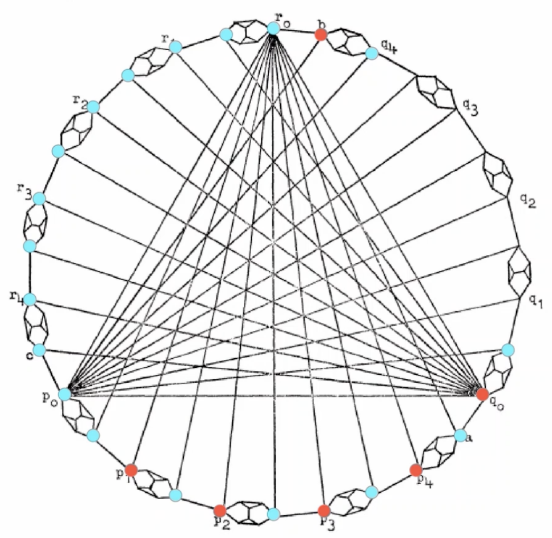

(This follows because we've seen from Clifton–Stairs that we can't have red on both sides of one of the little constructions.)

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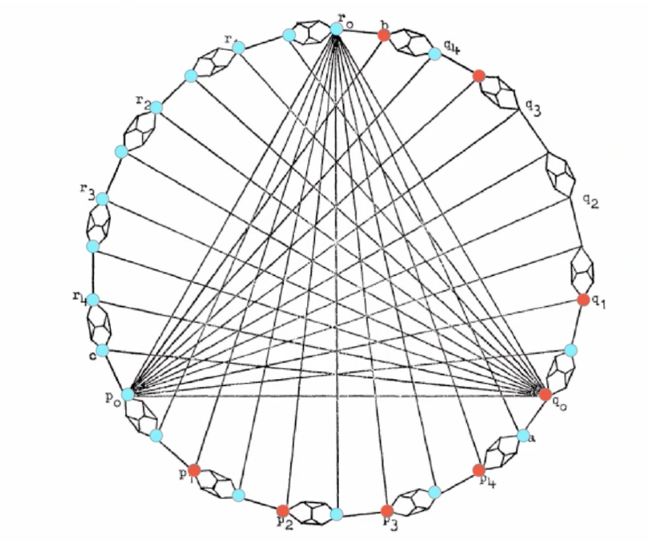

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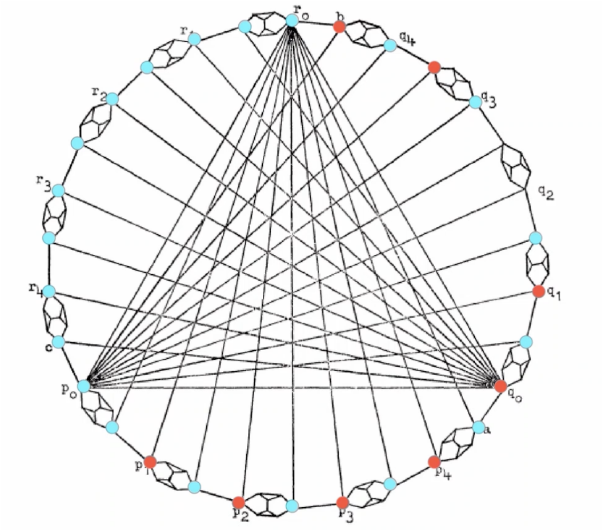

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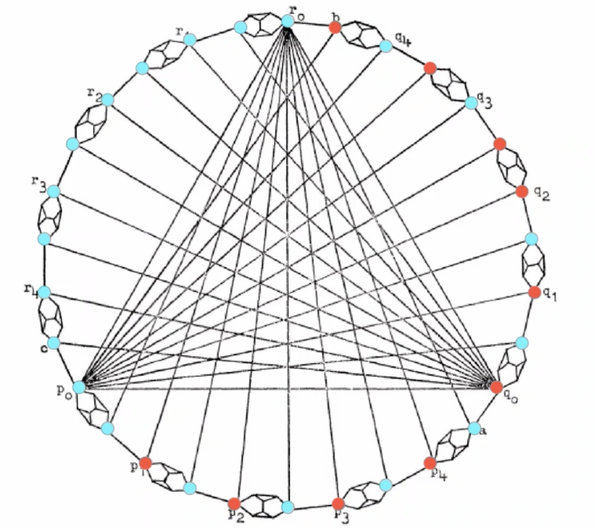

Contradiction, because we've already proven that if we've got one of the mini-diagrams then we can't have red on both sides!

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- Bell (1966) actually did something similar to Kochen & Specker (1967) earlier than them, but nobody realised until afterwards (hence now the 'BKS theorem').

Why triangles?

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Why triangles?

- In the BKS (and Clifton–Stairs) proof, we were considering triples of mutually orthogonal projectors.
- As such, we were working in dimension $d = 3$.
- But even in Hilbert spaces of larger dimension, we can always restrict to a 3D subspace and then run the same argument.
- So, the proof in fact suffices to show that one cannot have a non-contextual hidden variable theory in $d \geq 3$.

Later history of BKS-type results

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- E.g., Peres (1991) found a set of 33 with cubic symmetry.
- Penrose pointed out that Peres' set of 33 directions can be described as follows: take a cube and superimpose it with its 90-degree rotations about two perpendicular lines connecting its centre to the midpoints of an edge. Peres' directions point to the vertices and the centres of the faces and edges of the resulting set of three interpenetrating cubes. (Obviously...)

The Penrose cube

(Escher, *Waterfall*, 1961)

Today

Contextuality

The Bell–Kochen–Specker theorem introduced

The Clifton–Stairs state-dependent proof

Proof of the BKS theorem

The Klyachko–Can–Binicioglu–Shumovsky inequality

Significance of the BKS theorem

Gleason's theorem

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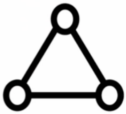
- I'll first present this, before discussing its significance.

The KCBS inequality



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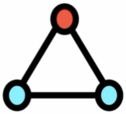

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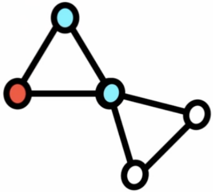

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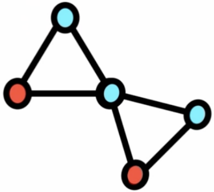

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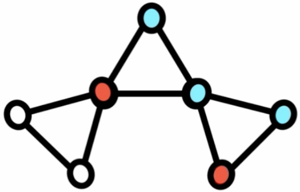

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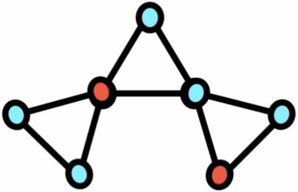

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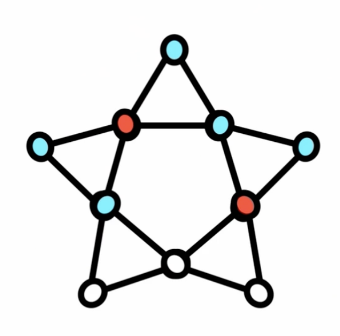

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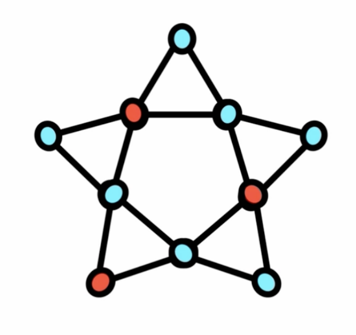

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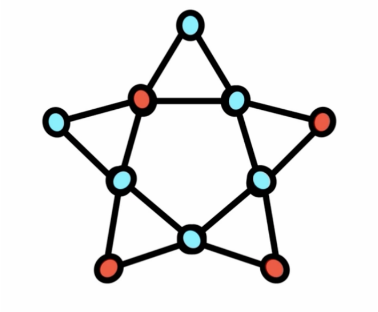

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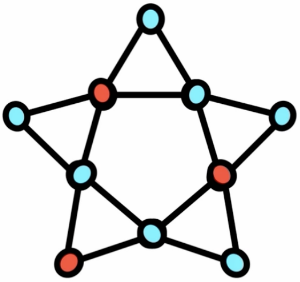

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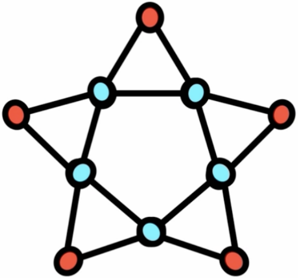

Main point: at least one of the outer points in the star must be red in a configuration like this.

The KCBS inequality

- Take any quantum state.

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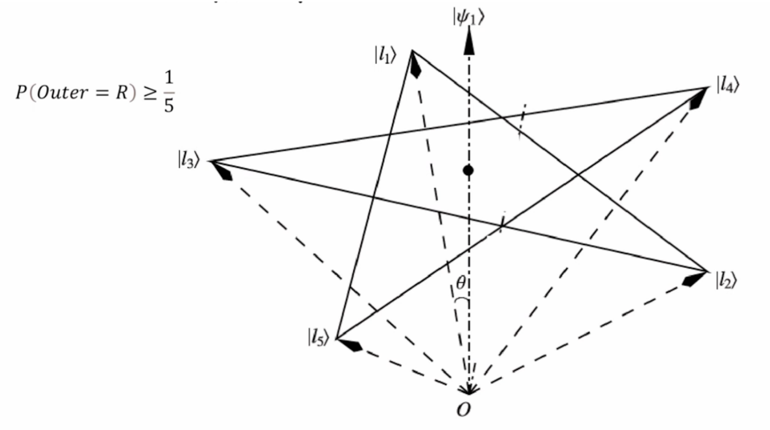
- From the five measurement bases (and measure it on that quantum state), Pr(Outer $= R$) $\geq \frac{1}{5}$.

The KCBS inequality

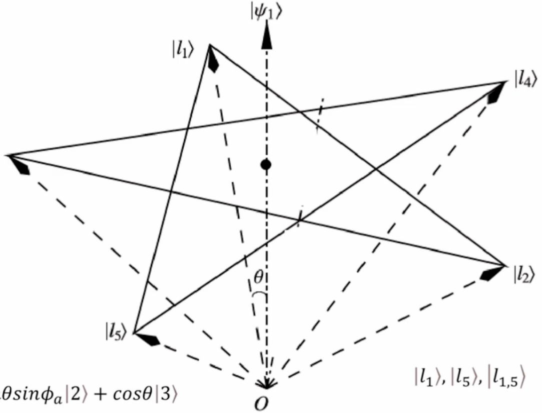
Now here comes quantum theory:



For any state you like, you can create this diagram around it.

The KCBS inequality

$$P(Outer = R) \geq \frac{1}{5}$$



$|l_1\rangle, |l_5\rangle, |l_{1,5}\rangle$

$$|l_a\rangle = sin\theta cos\phi_a|1\rangle + sin\theta sin\phi_a|2\rangle + cos\theta|3\rangle$$

$$|\langle 3|l_a\rangle|^2 = cos^2\theta = \frac{1}{\sqrt{5}}$$

$$cos\theta = \frac{1}{\sqrt[4]{5}} \qquad \phi_a = \frac{4}{5}\pi a$$

$$\left|\langle 3|l_{1,5}\rangle\right|^2 = 1 - \frac{2}{\sqrt{5}} = 0.1056\ldots < \frac{1}{5}$$

Experimental violation of the KCBS inequality

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  4. In light of this, do the two inequalities have the same role to play in 'experimental metaphysics'?
- The KCBS inequality at least helps us to *quantify* quantum non-contextuality (see e.g. Cabello 2013).

Today

Significance of the BKS theorem

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- Some authors, e.g. Acuña (2021), have argued that when one understands these connections, one sees that the significance of the BKS theorem has been substantially overblown.
- (But this is relatively new work and surely controversial!)

Von Neumann's theorem

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- Subsequently, authors such as Bub, Jammer, and Dieks have defended von Neumann's theorem, arguing that all it rules out are hidden variable theories whose ontic states are represented by Hermitian operators.
- Von Neumann's theorem is closely related to a famous theorem by Gleason (1957), which I will talk about in a minute. (See Acuña (2021), §4.)

BKS in light of von Neumann

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> *I argue that if we read the KST in the light of vNT and GT, we see that the former does not really force HVTs to be contextual—at least not in the usual ontological reading of 'contextual'—so it does not really oblige [hidden variable theorists] to make any important concessions in their basic motivation. As we will see, the reason is that the contextual-ontology reading of the KST crucially assumes that in HVTs Hermitian operators represent the theory's beables, but this is forbidden by vNT and GT.* (Acuña 2021, p. 3)

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So, for Acuña, the significance of the von Neumann and Gleason theorems is *much* greater than that of the BKS theorem.

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So, for Acuña, the significance of the von Neumann and Gleason theorems is *much* greater than that of the BKS theorem.

Note: This is a controversial claim which calls for detailed assessment!

Some further points

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- The BKS theorem is traditionally presented using PVMs. But it can be generalised to POVMs—see Spekkens (2005).

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Theorem (Gleason, 1957)

Let f be any function on projectors on a Hilbert space $\mathcal{H}$ of dimension $d > 2$ to the unit interval which is additive for any set of pairwise disjoint projectors on $\mathcal{H}$. Then there exists a unique density matrix $\hat{\rho}$ such that for any $\hat{P}$ on $\mathcal{H}$, $f(\hat{P}) = \mathrm{Tr}(\hat{\rho}\hat{P})$.

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- (The proof of Gleason's theorem is notoriously involved and I won't present it here.)
- One might think that Gleason's theorem counts as a derivation of the Born rule in unitary quantum mechanics—thereby (e.g.) solving (?) Everettians' worries about probability without the need to invoke decision theory etc.

Doubts about Gleason's theorem and quantum probabilities

However, as Saunders points out,

> *Gleason's theorem is a derivation of part of the Born rule, but of course it says nothing about 'measurements' or 'experiments'; nor, on reflection, is the premise of the theorem so clearly motivated.* (Saunders 2005, p. 213)

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Question: To what extent really is Gleason's theorem significant in the philosophy of quantum probabilities?

Gleason's theorem and the BKS theorem

- A direct consequence of Gleason's theorem is that when the Hilbert space $\mathcal{H}$ has dimension $d \geq 3$ then there does not exist a function f that assigns only the values 0 or 1 to projections $\hat{P} : \mathcal{H} \to \mathcal{H}$.

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- The BKS theorem is similar in spirit; indeed, the BKS theorem is provably a special case of Gleason's theorem.

A puzzle

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Question: What's going on?

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Next time: Ψ-ontic and Ψ-epistemic theories and the PBR theorem.

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